An Algorithm of Uniform Ultimate Boundedness for Switched Linear Systems*

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Abstract - *Uniform ultimate boundedness of a new class of switched linear systems is considered. A switched linear system in this class consists of m subsystems, and none of the individual subsystems need to be stabilizable. The switched linear system is shown to be uniformly ultimately bounded for any pre-given bound if the combination of controllable components of states for all subsystems covers the entire state space. An algorithm for the design of continuous controllers and the switching strategy is given to provide the design of continuous controllers and the switching strategy. Finally, the simulations show the validity of the result.*

Keywords: Hybrid dynamical systems, Switched systems, Controllability, Uniform ultimate boundedness

1 Introduction

Switched systems constitute an important class of hybrid dynamical systems. A typical switched system consists of several subsystems and a switching law specifying the active subsystem at each instant of time. Switched systems arise from many engineering applications. Typical examples of switched systems can be found, for example, in power systems[11], automated highway systems[9] and the control of the cart-pendulum system[5]. Switched systems also arise from the application of multiple controller which have been widely used in adaptive control[1], where a high-level, logic-based supervisor provides switching between a family of candidate controllers so as to achieve desired performance for the closed-loop systems.

In the study of switched systems, most works have been centralized on the problem of stability and stabilization[3,14]. A comprehensive survey on stability of switched systems was given in [2] with an update account of results and open problems. A switched system is stable under arbitrary switching laws if all subsystems have a common *Lyapunov* function. There are many results addressing conditions for the existence of a common *Lyapunov* function[6,8,10]. However, many switched systems do not have a common *Lyapunov* function. Therefore it has received a lot of attention to look for restricted class of switching laws under which switched systems are stable[7]. Single *Lyapunov* function technique and multiple *Lyapunov* function technique proposed by Branicky proved to be major tools in this aspect[7]. The later one seems to be even more powerful.

Though a number of methods have been proposed to guarantee stability of switched systems, there are still many switched systems, which are by no means stabilizable, might be ultimately bounded under appropriately designed switching laws. However, there are a few results on ultimate boundedness of switched systems have been reported by now. Zhang and Zhao considered the uniform ultimate boundedness of a class of switched linear systems in [13], An algorithm of uniform ultimate boundedness is given. In this paper, uniform ultimate boundedness of a new class of switched linear systems is considered. A switched linear system in this class consists of *m* subsystems, and none of the individual subsystems need to be stabilizable. The switched linear system is shown to be uniformly ultimately bounded for any pre-given bound if the combination of controllable components of states for all subsystems covers the entire state space. An algorithm for the design of continuous controllers and the switching strategy is given. Finally, the simulations show the validity of the result.

2 Main results

In the following discussion, $\|A\|$ denotes the Euclidean norm of the matrix A. We consider the following switched linear system

$$\dot{x}_c^i = A_{11}^i x_c^i + A_{12}^i x_{\bar{c}}^i + B_1^i u_i \quad (1a)$$

$$\dot{x}_{\bar{c}}^i = A_{22}^i x_{\bar{c}}^i \qquad i = 1,2,\cdots,m \quad (1b)$$

where i is the switching signal to be designed, $(x_c^i \quad x_{\bar{c}}^i)^{\mathrm{T}}$ is a partition of the n-dimensional vector x; x_c^i and $x_{\bar{c}}^i$ denote the controllable and uncontrollable components of x respectively. The switched system (1) is assumed to satisfy that every component x_j of vector $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in R^n$ appears in x_c^i for some *i*. In other words, the union of individual controllable subspaces of all subsystems covers the entire state space.

Our goal is to study uniform ultimate boundedness of the system (1). First of all, we give the precise definition of uniform ultimate boundedness for switched linear systems, which is a natural generalization of the concept for linear systems [8].

Definition 1 Suppose $i = i(t)$ is a switching law. The switched linear system

$$\dot{x} = A_i x, \quad i = 1,2,\cdots,k$$

is said to be uniformly ultimately bounded for the bound d, if for each $\delta > 0$, there exists $T = T(d,\delta) > 0$, which is independent of t_0, such that the trajectory $x(t)$ of the system starting from $x(t_0) = x_0$ satisfies $\|x(t)\| \le d$ for all $t \ge t_0 + T$ if $\|x_0\| < \delta$.

Before developing the main results of uniform ultimate boundedness of the switched system (1), we analyze what the special structure of the system (1) can bring us.

Theorem 1 Given the system

$$\dot{x}_c = A_{11} x_c + A_{12} x_{\bar{c}} + Bu \quad (2a)$$

$$\dot{x}_{\bar{c}} = A_{22} x_{\bar{c}} \quad (2b)$$

where (2a) is a controllable subsystem. Then for any $M > 0$, there exists a constant $T > 0$, such that when $T < M$, the following results hold.

If (2a) is a 1st order subsystem or 2nd order subsystem, then there exist feedback $u = Kx_c$, such that the states of the closed-loop system with any initial values $x_c(0)$, $x_{\bar{c}}(0)$ and $x_c(0) \ne 0$ satisfy:

$$\|x_c(T)\| + \|x_{\bar{c}}(T)\| \le \|x_c(0)\| + \|x_{\bar{c}}(0)\| - \frac{1}{4}\|x_c(0)\| \quad (3)$$

Proof: For (2a) is a controllable 1st order subsystem or 2nd order subsystem. Let the controller be $u = Kx_c$, and $\tilde{A}_{11} = A_{11} + BK$, then all the eigenvalues of the matrix $\tilde{A}_{11}$ have negative real part, there exists $\alpha > 0$, such that $\left\|e^{\tilde{A}_{11}t}\right\| < e^{-\alpha t}$. The states of the closed-loop system (2) can be expressed as

$$x_c(t) = e^{\tilde{A}_{11}t} x_c(0) + \int_0^t e^{\tilde{A}_{11}(t-\tau)} A_{12} e^{A_{22}\tau} d\tau \, x_{\bar{c}}(0) \quad (4a)$$

$$x_{\bar{c}}(t) = e^{A_{22}t} x_{\bar{c}}(0) \quad (4b)$$

then

$$\|x_c(T)\| \le \left\|e^{\tilde{A}_{11}T} x_c(0)\right\| + \left\|\int_0^T e^{\tilde{A}_{11}(T-\tau)} A_{12} e^{A_{22}\tau} d\tau \, x_{\bar{c}}(0)\right\| \quad (5a)$$

$$\|x_{\bar{c}}(T)\| \le \left\|e^{A_{22}T}\right\| \|x_{\bar{c}}(0)\| \quad (5b)$$

Let $N_t = \sup_{s\in[0,t]} \left\|e^{A_{22}s}\right\|$. It is obvious that N_t is monotonously increasingly continuous and satisfies $N_0 = 1$. Then the inequality (5) gives rise to

$$\|x_c(T)\| \le \|e^{\tilde{A}_{11}T}\| \|x_c(0)\| + N_T \|A_{12}\| \times \left\| \int_0^T \|e^{\tilde{A}_{11}(T-\tau)}\| d\tau \, \|x_{\bar{c}}(0)\| \right\|$$

$$\le e^{-\alpha T} \|x_c(0)\| + N_T \|A_{12}\| \cdot \frac{1}{\alpha}(1-e^{-\alpha T}) \cdot \|x_{\bar{c}}(0)\| \quad (6a)$$

$$\|x_{\bar{c}}(T)\| \le \|e^{A_{22}T}\| \cdot \|x_{\bar{c}}(0)\| \le N_T \cdot \|x_{\bar{c}}(0)\| \quad (6b)$$

Then we have

$$\|x_c(T)\| + \|x_{\bar{c}}(T)\| \le N_T \cdot \|x_{\bar{c}}(0)\| + e^{-\alpha T}\|x_c(0)\| + \frac{1}{\alpha} N_T \|A_{21}\| (1-e^{-\alpha T}) \|x_{\bar{c}}(0)\|$$

$$\le \|x_c(0)\| + \|x_{\bar{c}}(0)\| + [e^{-\alpha T} - 1]\|x_c(0)\| + [N_T + \frac{1}{\alpha} N_T \|A_{12}\| (1-e^{-\alpha T}) - 1]\|x_{\bar{c}}(0)\| \quad (7)$$

Now select α, T such that

$$(e^{-\alpha T} - 1)\|x_c(0)\| + (N_T + \frac{1}{\alpha} N_T \|A_{12}\| \times (1-e^{-\alpha T}) - 1]\|x_{\bar{c}}(0)\| \le -\frac{1}{4}\|x_c(0)\| \quad (8)$$

then the theorem 1 holds.

Remark 1 According to geometric control theory, when a k-dimensional subspace V is the invariant subspace of states matrix A, and subspace V includes the image space of matrix B, then the systems can be expressed in the form of (2), which have the same structure as that of any subsystem of (1). (refer to [4])

Now, we give the main theorem of uniform ultimate boundedness of the switched system (1):

Theorem 2 Suppose

i) If (1a) is a 1st order controllable system or a 2nd order controllable system;

ii) Each component x_j of n-dimensional vector x is contained in x_c^i for some *i*.

Then there exist feedback $u_i = K_i x$ and a switching law such that the closed-loop system of the switched system (1) is uniformly ultimately bounded for any pre-given bound *d*.

Proof: In order to prove the theorem we give an algorithm that provides the design of continuous controllers and the switching law.

Algorithm

Let $x(0)$ be any initial condition.

Step 1: Set $k = 0$, $t_k = 0$.

Step 2: Calculate $\max_i \|x_c^i(t_k)\|$. Let i be the minimum subscript at which the maximum is reached. Then, the *i*-th subsystem is activated until the time t_{k+1} determined by Step 3.

Step 3: According to the proof of theorem 1, calculate the feedback $u_i = K_i x_c^i$ and t_{k+1}, such that

$$\|x_c^i(t_{k+1})\| + \|x_{\bar{c}}^i(t_{k+1})\| \le \|x_c^i(t_k)\| + \|x_{\bar{c}}^i(t_k)\| - \frac{1}{4}\|x_c^i(t_k)\| \quad (9)$$

Step 4: Set $k = k+1$, go back to Step 2

The algorithm generates the time sequence $\{t_k\}$ at which the switching occurs. From Theorem 1 and step 2 of the algorithm, we know that there exist $\delta > 0$ and a subsequence t_{k_j} of $\{t_k\}$, such that $t_{k_{j+1}} - t_{k_j} \ge \delta$, which implies that only a finite number of switches occur in any finite time. Furthermore, we can choose a small constant $\rho > 0$ such that $t_{k+1} - t_k \ge \rho$ for all *k*. This is because the system (1a) is controllable and thus we can design the linear feedback $u_i = K_i x$ such that x_c^i converges quickly enough to guarantee (9) for some

t_{k+1} satisfying $t_{k+1} - t_k \geq \rho$. Noticing (9) we know that the sequence $\{\|x_c^i(t_{k+1})\| + \|x_{\bar{c}}^i(t_{k+1})\|\}$ is strictly decreasing, which obviously in turn implies $x(t_k) \to 0$. Now, it is easy to see that for any given bound d and an initial point $x(0)$ in any known bounded set Ω, there exists T, which may depend on d and Ω, such that $\|x(t)\| \leq d$ for all $t \geq t_0 + T$, and thus the uniform ultimate boundedness follows.

3 Simulation

We consider the switched linear system

$$\dot{x} = A_i x + B_i u_i \qquad i = 1,2 \qquad (10)$$

where the first and second subsystems are given respectively by

$$\begin{aligned} \dot{x}_1^1 &= -3x_1^1 + x_2^1 + u_1 \\ \dot{x}_2^1 &= 3x_2^1 \end{aligned} \qquad (11a)$$

and

$$\begin{aligned} \dot{x}_1^2 &= x_1^2 \\ \dot{x}_2^2 &= 3x_1^2 - 4x_2^2 + u_2 \end{aligned} \qquad (11b)$$

The state responds of the two subsystems starting from $x(0) = (-3,\ 4)^{\mathrm{T}}$ are shown in Figure 1 and Figure 2, which indicate that neither of the two subsystems is stable.

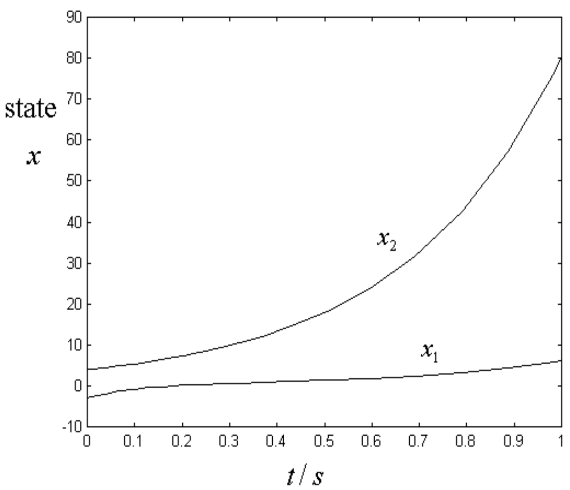

Figure 1 The states respond of the first subsystem

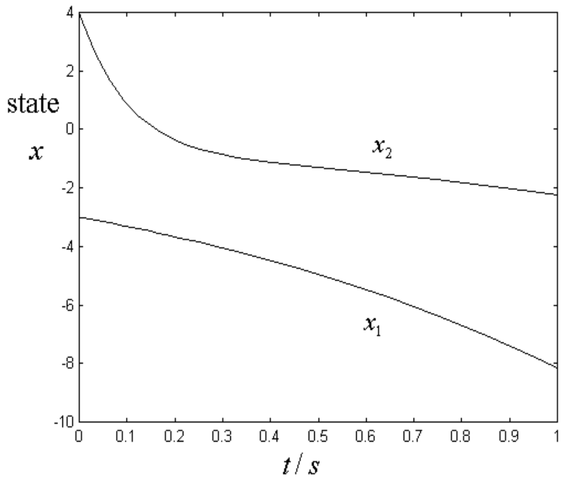

Figure 2 The states respond of the second subsystem

It is easy to see that x_1-substate of the first subsystem and x_2-substate of the second subsystem are controllable. We design $u_1 = -7x_1^1$ and $u_2 = -6x_2^2$. The switching law determined by Step 2 and Step 3 of the algorithm gives $t_0 = 0$, $t_1 = 0.13$, $t_2 = 0.37$, $t_3 = 0.56$, $t_4 = 0.77$, $t_5 = 1$, $t_6 = 1.5, \cdots$. The initial condition is again $x(0) = (-3,\ 4)^{\mathrm{T}}$. The simulation results are shown in Figure 3 and Figure 4, from which we see that the system is uniformly ultimately bounded.

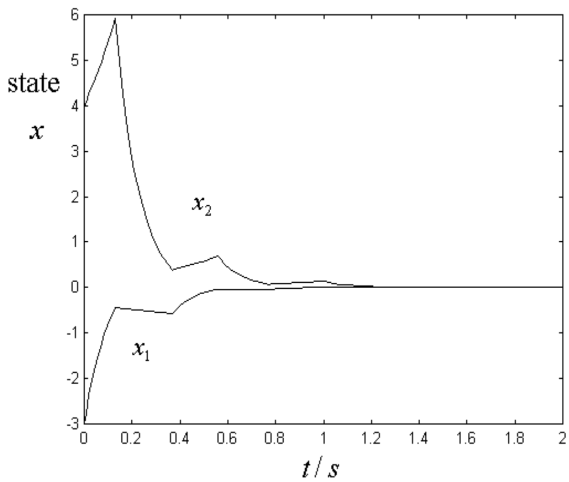

Figure 3 The states respond of the switched system

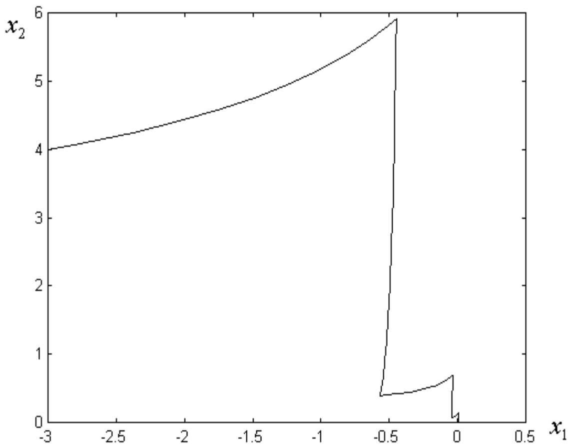

Figure 4 The phase plane of the switched systems

4 Concluding remark

A new class of switched linear systems have been shown to be uniformly ultimately bounded for any pre-given bound as long as the union of all controllable subspaces of subsystems covers the entire state space. The continuous state feedback controllers for subsystems have been designed according to linear control theory, which ensure the decay of states to certain extent during the time interval when a subsystem is activated. The design process is applicable to switched systems consisting of no stable subsystems.

Acknowledgements

The work was supported by the National Science Foundation of China under Grant 60274046.

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